# CHAPTER 3

# Matrices

## Special Type of Matrices

1. **Row Matrix (Row vector):** $A = [a_{11}, a_{12}, \ldots a_{1n}]$ i.e., row matrix has exactly one row.
2. **Column Matrix (Column vector):** $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$ i.e., column matrix has exactly one column.
3. **Zero or Null Matrix:** $(A = O_{m\times n})$, an $m \times n$ matrix whose all entries are zero.
4. **Horizontal Matrix:** A matrix of order $m \times n$ is a horizontal matrix if $n > m$.
5. **Vertical Matrix:** A matrix of order $m \times n$ is a vertical matrix if $m > n$.
6. **Square Matrix:** (Order $n$) if number of rows = number of column, then matrix is a square matrix.

**Key Note**

- The pair of elements $a_{ij}$ and $a_{ji}$ are called Conjugate Elements.
- The elements $a_{11}, a_{22}, a_{33}, \ldots a_{mm}$ are called Diagonal Elements. the line along which the diagonal elements lie is called "Principal or Leading diagonal." The quantity $\Sigma a_{\text{ii}}$ = trace of the matrix written as, $t_r(A)$.

7. **Unit/Identity Matrix:** A square matrix, in which every non-diagonal element is zero and every diagonal element is 1, is called unit matrix or an identity matrix,

   i.e. $a_{ij} = \begin{cases} 0, \text{ when } i \neq j \\ 1, \text{ when } i = j \end{cases}$

8. **Upper Triangular Matrix:** A square matrix $A = [a_{ij}]_{n\times n}$ is called a upper triangular matrix, if $a_{ij} = 0, \forall\ i > j$.
9. **Lower Triangular Matrix:** A square matrix $A = [a_{ij}]_{n\times n}$ is called a lower triangular matrix, if $a_{ij} = 0, \forall\ i < j$.
10. **Submatrix:** A matrix which is obtained from a given matrix by deleting any number of rows or columns or both is called a submatrix of the given matrix.
11. **Equal Matrices:** Two matrices $A$ and $B$ are said to be equal, if both having same order and corresponding elements of the matrices are equal.
12. **Principal Diagonal of a Matrix:** In a square matrix, the diagonal from the first element of the first row to the last element of the last row is called the principal diagonal of a matrix.

    e.g. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 6 & 5 \\ 1 & 1 & 2 \end{bmatrix}$, the principal diagonal of $A$ is 1, 6, 2.

13. **Singular Matrix:** A square matrix $A$ is said to be singular matrix, if determinant of $A$ denoted by det $(A)$ or $|A|$ is zero, i.e. $|A| = 0$, otherwise it is a non-singular matrix.

## Equality of Matrices

Let $A = [a_{ij}]$ & $B = [b_{ij}]$ are equal if,

1. Both have the same order.
2. $a_{ij} = b_{ij}$ for each pair of $i$ & $j$.

## Algebra of Matrices

**Addition:** $A + B = [a_{ij} + b_{ij}]$ where $A$ & $B$ are of the same order.

1. **Addition of matrices is commutative:** $A + B = B + A$.
2. **Matrix addition is associative:** $(A + B) + C = A + (B + C)$.

## Multiplication of a Matrix By a Scalar

If A = $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, then $kA = \begin{bmatrix} ka & kb & kc \\ kb & kc & ka \\ kc & ka & kb \end{bmatrix}$

## Multiplication of Matrices (Row by Column)

Let $A$ be a matrix of order $m \times n$ and $B$ be a matrix of order $p \times q$ then the matrix multiplication $AB$ is possible if and only if $n = p$.

Let $A_{m \times n} = [a_{ij}]$ and $B_{n \times p} = [b_{ij}]$, then order of $AB$ is $m \times p$ and

$$\boxed{(AB)_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}}$$

## Characteristic Equation

Let $A$ be a square matrix. Then the polynomial $| A - xI |$ is called as characteristic polynomial of $A$ & the equation $| A - xI | = 0$ is called characteristic equation of $A$.

## Properties of Matrix Multiplication

**1.** If $A$ and $B$ are two matrices such that

(i) $AB = BA$ then $A$ and $B$ are said to commute

(ii) $AB = -BA$ then $A$ and $B$ are said to anticommute

**2. Matrix Multiplication is Associative:** If $A$, $B$ & $C$ are conformable for the product $AB$ & $BC$, then $(AB)C = A(BC)$.

**3. Distributivity:** $\left.\begin{matrix} A(B+C) = AB + AC \\ (A+B)C = AC + BC \end{matrix}\right]$, provided $A$, $B$ and $C$ are conformable for respective products.

## Positive Integral Powers of a Square Matrix

**1.** $A^m A^n = A^{m+n}$

**2.** $(A^m)^n = A^{mn} = (A^n)^m$

**3.** $I^m = I$ $m, n \in N$

## Orthogonal Matrix

A square matrix is said to be orthogonal matrix if $AA^T = I$.

**Key Note**

- The determinant value of orthogonal matrix is either 1 or –1.
  Hence orthogonal matrix is always invertible.
- $AA^T = I = A^T A$. Hence $A^{-1} = A^T$.

## Some Square Matrices

**1. Idempotent Matrix:** A square matrix is idempotent provided $A^2 = A$. For idempotent matrix note the following:

**(a)** $A^n = A \ \forall \ n \geq 2, n \in N$.

**(b)** determinant value of idempotent matrix is either 0 or 1.

**(c)** If idempotent matrix is invertible then its inverse will be identity matrix i.e. $I$.

**2. Periodic Matrix:** A square matrix which satisfies the relation $A^{K+1} = A$, for some positive integer $K$, is a periodic matrix. The period of the matrix is the least value of $K$ for which this holds true.

Note that period of an idempotent matrix is 1.

**3. Nilpotent Matrix:** A square matrix is said to be nilpotent matrix of order $m$, $m \in N$, if $A^m = O$, $A^{m-1} \neq O$.

Note that a nilpotent matrix will not be invertible.

**4. Involutary Matrix:** If $A^2 = I$, the matrix is said to be an involutary matrix.

Note that $A = A^{-1}$ for an involutary matrix.

**5.** If $A$ and $B$ are square matrices of same order and $AB = BA$ then

$(A + B)^n = {}^nC_0 A^n + {}^nC_1 A^{n-1} B + {}^nC_2 A^{n-2} B^2 + \ldots + {}^nC_n B^n$.

## Transpose of a Matrix (Changing Rows & Columns)

Let $A$ be any matrix of order $m \times n$. Then $A^T$ or $A' = [a_{ij}]$ for $1 \leq i \leq n$ & $1 \leq j \leq m$ of order $n \times m$.

## Properties of Transpose

If $A^T$ & $B^T$ denote the transpose of $A$ and $B$

**1.** $(A + B)^T = A^T + B^T$ ; note that $A$ & $B$ have the same order.

**2.** $(A\,B)^T = B^T A^T$ (Reversal law) $A$ & $B$ are conformable for matrix product $AB$

**3.** $(A^T)^T = A$

**4.** $(kA)^T = kA^T$, where is a scalar.

**General:** $(A_1 \cdot A_2, \ldots A_n)^T = A^T_n \cdot \ldots A^T_2 \cdot A^T_1$ (reversal law for transpose)

## Symmetric & Skew Symmetric Matrix

**1. Symmetric matrix:** For symmetric matrix $A = A^T$.

**Note:** Maximum number of distinct entries in any symmetric matrix of order $n$ is $\dfrac{n(n+1)}{2}$.

**2. Skew symmetric matrix:** Square matrix $A = [a_{ij}]$ is said to be skew symmetric if $a_{ij} = -a_{ji} \ \forall \ i$ & $j$. Hence if $A$ is skew symmetric, then $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0 \ \forall \ i$.

Thus the diagonal elements of a skew square matrix are all zero, but not the converse.

For a skew symmetric matrix $A = -A^T$.

**3.** Properties of symmetric & skew symmetric matrix:

**(a)** Let $A$ be any square matrix then, $A + A^T$ is a symmetric matrix and $A - A^T$ is a skew symmetric matrix.

**(b)** The sum of two symmetric matrix is a symmetric matrix and the sum of two skew symmetric matrix is a skew symmetric matrix.

**(c)** If A & B are symmetric matrices then,

(i) $AB + BA$ is a symmetric matrix.

(ii) $AB - BA$ is a skew symmetric matrix.

**4.** Every square matrix can be uniquely expressed as a sum or difference of a symmetric and a skew symmetric matrix.

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew symmetric}}$$

and $$A = \frac{1}{2}(A^T + A) - \frac{1}{2}(A^T - A)$$

## Adjoint of a Square Matrix

Let $A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a square matrix and let the matrix formed by the cofactors of $[a_{ij}]$ in determinant $|A|$ is

$\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$. Then (adj $A$) $= \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$.

**Key Note**

If $A$ be a square matrix of order $n$, then

**1.** $A(\text{adj } A) = |A|\, I_n = (\text{adj } A)\, .\, A$

**2.** $|\text{adj } A| = |A|^{n-1}$

**3.** $\text{adj}(\text{adj } A) = |A|^{n-2} A$

**4.** $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

**5.** $\text{adj }(AB) = (\text{adj } B)\,(\text{adj } A)$

**6.** $\text{adj }(KA) = K^{n-1}\,(\text{adj } A)$, where $K$ is a scalar

## Inverse of a Matrix (Reciprocal Matrix)

A square matrix $A$ (non singular) said to be invertible, if there exists a matrix $B$ such that, $AB = I = BA$.

$B$ is called the inverse (reciprocal) of $A$ and is denoted by $A^{-1}$. Thus

$$A^{-1} = B \quad \Leftrightarrow \quad AB = I = BA$$

We have, $A \cdot (\text{adj } A) = |A|\, I_n$

$$A^{-1} \cdot A(\text{adj } A) = A^{-1}\, I_n\, |A|$$

$$I_n\, (\text{adj } A) = A^{-1}\, |A|\, I_n$$

$$\therefore \quad A^{-1} = \frac{(\text{adj } A)}{|A|}$$

**Note:** The necessary and sufficient condition for a square matrix $A$ to be invertible is that $|A| \neq 0$.

**Theorem:** If $A$ and $B$ are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

**Key Note**

- If $A$ be an invertible matrix, then $A^T$ is also invertible and $(A^T)^{-1} = (A^{-1})^T$.
- If $A$ is invertible,
  (a) $(A^{-1})^{-1} = A$
  (b) $(A^k)^{-1} = (A^{-1})^k = A^{-k}$ ; $k \in N$

## System of Equation and Criteria for Consistency

**Gauss - Jordan Method**

**Example:**

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$\Rightarrow \begin{bmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\Rightarrow \quad AX = B \quad \Rightarrow \quad A^{-1}AX = A^{-1}B$$

$$\Rightarrow \quad X = A^{-1}\, B = \frac{\text{Adj } A}{|A|} \cdot B$$

**Key Note**

- If $|A| \neq 0$, system is consistent having unique solution.
- If $|A| \neq 0$ and (adj $A$) $\cdot$ $B \neq O$ (Null matrix), system is consistent having unique non-trivial solution.
- If $|A| \neq 0$ and (adj $A$) $\cdot$ $B = O$ (Null matrix), system is consistent having trivial solution.
- If $|A| = 0$, then

**Matrix Method Fails**

If (adj $A$) $\cdot$ $B = O$ (null matrix) → Consistent (infinite solutions)

If (adj $A$) $\cdot$ $B \neq O$ → Inconsistent (no solutions)